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QUASI-LOCALIZATIONS OF $\mathbb{Z}$

BY

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ABSTRACT

Motivated by the categorical notion of localizations applied to the quasi-category of abelian groups, we call a homomorphism $\alpha: A \to B$ a quasi-localization of abelian groups if for each $\varphi \in Hom(A, B)$ there is an $n \in \mathbb{N}$ and a unique $\psi \in End(B)$ such that $n\varphi = \psi \circ \alpha$. In this case we call B a quasi-localization of A. In this paper we investigate quasi-localizations of the integers $\mathbb{Z}$. While it is well-known that localizations of $\mathbb{Z}$ are just the E-rings, quasi-localizations of $\mathbb{Z}$ are much more abundant; an injection $\alpha: \mathbb{Z} \to M$ with M torsion-free, is a quasi-localization if and only if, for $R = End(M)$, one has $R \subseteq M \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} R$. We call R the ring of the quasi-localization M. Some old results due to Zassenhaus and Butler show that all rings with free additive groups of finite rank are indeed rings of quasi-localizations of $\mathbb{Z}$. We will extend this result and show that there are also rings of infinite rank with this property. While there are many realization results of rings R as endomorphism rings of torsion-free abelian groups M in the literature, the group M is usually not contained in the divisible hull of R^+, as is required here. We will use a particular case of a category of left R-modules M with a distinguished family $\mathcal{F}$ of submodules and thus $End(M, \mathcal{F}) = \{\psi \in End(M) : \psi(X) \subseteq X \text{ for all } X \in \mathcal{F}\}$. We will restrict our discussion to the case $M = R$ such that $End(R, \mathcal{F}) = R$, and in this case we call the family $\mathcal{F}$ of left ideals E-forcing, not to be confused with the notion of forcing in set theory. We will provide many examples of quasi-localizations M of $\mathbb{Z}$, among them those of infinite rank as well as matrix rings for various rings of finite rank.

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1. Introduction

Let $\mathcal{C}$ be any category, A, B objects in $\mathcal{C}$, and $\alpha\colon A \to B$ a morphism in $\mathcal{C}$. Then α is called a localization of A if for any morphism $\varphi\colon A \to B$ there is a **unique** morphism $\psi\colon B \to B$ such that $\varphi = \psi \circ \alpha$. Localizations are an important notion in category theory and elsewhere. Recently, localizations in the category of abelian groups were studied in [4], [6], [7], [11], [15] and in other articles. We will use the following standard identifications (see e.g. [6], [7], [9], [10]). If M is a faithful left R-module, then any $r \in R$ will be identified with the scalar multiplication by r on the left. Thus $r \in End(M)$ and similarly $R \subseteq End(M)$. It is well-known, that if $\alpha\colon \mathbb{Z} \to B$ is a localization of abelian groups, then B is the additive group of an E-ring R, i.e. if R^+ is the additive group of R then $End(R^+) = R$, cf. [4], [13] for example. E-rings show up frequently in the theory of abelian groups, especially in the theory of torsion-free abelian groups of finite rank. See [17] or [13] for a nice survey article on E-rings and their generalizations. A frequently utilized tool in the investigation of torsion-free abelian groups of finite rank is their quasi-category, i.e. the objects are just all those groups, but $\mathbb{Q}Hom(A, B)$ is the set of morphisms from A to B. Considering localizations in that category naturally leads to the following

Definition 1: A homomorphism $\alpha\colon A \to B$ is a quasi-localization of A if for all homomorphisms $\varphi\colon A \to B$ there exists some natural number n and a **unique** homomorphism $\psi\colon B \to B$ such that $n\varphi = \psi \circ \alpha$. (We will usually restrict our attention to the case where α is injective.)

Note that this definition makes sense for torsion-free abelian groups of any rank. All groups in this paper are torsion-free unless stated otherwise. In this paper we will pursue the natural question: What are the quasi-localizations of $\mathbb{Z}$? Here is the answer in a nutshell:

- Let $\alpha\colon \mathbb{Z} \to M$ be an injective homomorphism and M a torsion-free abelian group. Then α is a quasi-localization if and only if $R \subseteq M \subseteq \mathbb{Q}R$ as left R-modules and $R = End(M)$. In this case $\alpha(1) = 1 \in R$ and we call R **the ring of the quasi-localization** of $\mathbb{Z}$.

This shows that quasi-localizations of $\mathbb{Z}$ are much more abundant than the localizations of $\mathbb{Z}$, i.e. the E-rings. In 1967 H. Zassenhaus [18] proved that if R is any unital ring with additive free abelian group of finite rank, then there exists $R \subseteq M \subseteq \mathbb{Q}R$ such that $End(M) = R$.

Recall that A. L. S. Corner [5] found examples of torsion-free rings R of rank n (R^+ is p-local) such that R *is not* the endomorphism ring of any torsion-free group of rank less than $2n$. This indicates that the question of which torsion-free rings of finite rank are rings of quasi-localizations of $\mathbb{Z}$ might be quite difficult to answer. Reid and Vinsonhaler [14] generalized the result in [3] by replacing the base ring $\mathbb{Z}$ with a subring K of an algebraic number field.

In our pursuit to find new torsion-free rings of finite rank for which a Zassenhaus/Butler type result holds, we look at matrix rings, employ a result in [1], and show:

- Let S be any torsion-free ring of finite rank such that the $\mathbb{Q}$-algebra $\mathbb{Q}S$ is generated by a set $\{1, \gamma_2, \ldots, \gamma_{m-1}\}$ as an algebra. If S^+ is p-reduced for at least $2m+6$ distinct primes p, then the matrix ring $R = Mat_{(2m)\times(2m)}(S)$ is the ring of a quasi-localization M of $\mathbb{Z}$. If S happens to be an E-ring, that number can be reduced to $2m+1$. In particular, $R = Mat_{n\times n}(\mathbb{Z})$ is, for any $n \in \mathbb{N}$, the ring of a quasi-localization of $\mathbb{Z}$. In any case, M/R is a direct sum of copies of $\mathbb{Z}(p^\infty)$ for finitely many primes p.

As far as we know, there are no examples in the literature of any torsion-free rings R of *infinite* rank that are the rings of quasi-localizations M of $\mathbb{Z}$. The main purposes of this paper is to find such rings, R, for which such an M exists and also to say some more about M/R for some special rings R of finite rank.

In order to be able to work inside the ring R we introduce the notion of an E-forcing family for a ring R:

Definition 2: Let R be a ring, $1 \in R$, which is also an algebra over a (commutative) ring S. Let $\mathcal{F}$ be a family of some (algebra) left ideals of R. Then $\mathcal{F}$ is called an E-**forcing family** of left ideals of R if

$End(R,\mathcal{F}) = \{\varphi \in End_S(R^+) : \varphi(X) \subseteq X \text{ for all } X \in \mathcal{F}\} = R$. (If $S = \mathbb{Z}$, then we drop the subscript S and set $End_S(R^+) = End(R^+)$.)

Of course, if R is an E-ring, then $\varnothing$ is an E-forcing family, and any ring R has an E-forcing family if and only if the set *of all* left ideals is an E-forcing family. But since those families are used to construct quasi-localizations M, one wants the family $\mathcal{F}$ to be as small and "nice" as possible. Large E-rings were first constructed in [8]. Simson's book [16] is an excellent source for information on the theory of modules with distinguished submodules.

- Let S be any unital ring and $R = Mat_{n\times n}(S)$ the S-algebra of $n \times n$-matrices over S. Then R has an E-forcing family of $n+1$ left ideals that are S-summands of R. As a consequence, $R = Mat_{n\times n}(\mathbb{Z})$ is the ring of a quasilocalization M of $\mathbb{Z}$ such that $t_p(M/R) \neq 0$ for only $n+1$ primes p and is torsion divisible.

We will employ a Black Box construction to show that arbitrarily large quasi-localizations of $\mathbb{Z}$ exist that are not E-rings:

- Let $\lambda = (\mu^{\aleph_0})^+$ for some infinite cardinal μ and $\mathbb{Z}[X]$ the ring of integer polynomials in λ-many variables. Then $\mathbb{Z}[X]$ is the ring of quasi-localizations of $\mathbb{Z}$.

A. L. S. Corner, cf. [9, page 145], presented a countable, torsion-free ring R_C with many idempotents such that whenever $End(M) = R_C$, then M is super-decomposable, i.e. M has no indecomposable summands. We use our approach to show:

- R_C has an E-forcing family and is the ring of a quasi-localization M of $\mathbb{Z}$, i.e. M is super-decomposable.

Finally, we consider the nice class of rings R of algebraic integers of Galois extensions of $\mathbb{Q}$ and show that these rings are rings of quasi-localizations M of $\mathbb{Z}$, such that $p(t_p(M/R)) = 0$ for all primes p.

2. First Results

We view any abelian group B as a left module over the ring $R = End(B)$ and describe the quasi-localizations of $\mathbb{Z}$ as follows.

Theorem 1: *Let B be a torsion-free group and $\alpha\colon \mathbb{Z} \to B$ an injective homomorphism. Then α is a quasi-localization if and only if for $R = End(B)$, we have $\alpha(1) = 1 \in R$ with $R \subseteq B \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} R$ as R-modules. (In this case we call R the ring of the quasi-localization α of $\mathbb{Z}$.)*

Proof: Suppose that $\alpha\colon \mathbb{Z} \to B$ is a quasi-localization. Then, for each $\varphi \in Hom(\mathbb{Z}, B)$ there is a least $n(\varphi) \in \mathbb{N}$ such that there is a unique $\psi \in End(B)$ the satisfies $n(\varphi)\varphi = \psi \circ \alpha$. Let $R = End(B)$. Then $B =_R B$ is a unital left R-module. Let $B_1 = \{b \in B : n(b^*) = 1\}$, where $b^* \in Hom(\mathbb{Z}, B)$ is defined by $b^*(1) = b$. Note that $n(n(\varphi)\varphi) = 1$ for all $\varphi \in Hom(\mathbb{Z}, B)$, which implies that $n(b^*)b \in B_1$ for all $b \in B$, i.e. B/B_1 is torsion. Since B is torsion-free we infer $B \subseteq \mathbb{Q}\otimes_{\mathbb{Z}}B_1 = \mathbb{Q}B_1$ for short. We will show that $B_1 = R(\alpha(1))$. If $b \in B_1$ then there is some $\psi \in R$ with $b = b^*(1) = \psi(\alpha(1))$ and we have that $B_1 \subseteq R(\alpha(1))$. If, on the other hand, $\psi(\alpha(1)) \in R(\alpha(1))$, then $[\psi(\alpha(1))]^* = \psi\circ\alpha$ and $\psi(\alpha(1)) \in B_1$ follows. If $\psi \in R$ with $\psi(\alpha(1)) = 0$, then $0^* = \psi\circ\alpha$ and $\psi = 0$ follows by uniqueness. This shows that one can identify B_1 with R and we have $R \subseteq B \subseteq \mathbb{Q}R$. Note that $[\alpha(1)]^* = id_B \circ \alpha$ and by uniqueness, $\alpha(1) = 1 \in R$.

To show the converse, let $\alpha\colon \mathbb{Z} \to B$, $R \subseteq B \subseteq \mathbb{Q}R$ with $\alpha(1) = 1 \in R$. Let $\varphi = b^* \in Hom(\mathbb{Z}, B)$ for some $b \in B$. Then there exists some $n \in \mathbb{N}$ such that $nb = r \in R$ and it follows that $nb = (n\varphi)(1) = (r \circ \alpha)(1) = r\alpha(1) = r1$. Moreover, $r \in R = End(B)$ is unique with that property. ∎

Zassenhaus [18] showed that if R is a ring with finite rank free abelian R^+, then there does indeed exist $R \subseteq B \subseteq \mathbb{Q}R$ such that $End(B) = R$. This result was generalized by Butler [3] to finite rank rings R such that R^+ is locally free at each prime p. Note that this shows that the ring R of a quasi-localization of $\mathbb{Z}$ need not be an E-ring and not even commutative! On the other hand, Corner [5] gave examples of torsion-free rings R of rank n, such that R is *not* the endomorphism ring of any abelian group of rank less than $2n$. This shows that some finite rank torsion-free rings *are* rings of quasi-localizations of $\mathbb{Z}$ and some *are not*.

PROPOSITION 1: *Let $R \subseteq B \subseteq \mathbb{Q}R$ be a quasi-localization of $\mathbb{Z}$ with R^+ torsion-free. If the torsion group B/R is bounded, then R is an E-ring.*

Proof: Let $\varphi \in End(R^+)$ and assume $nB \subseteq R \subseteq B$. Define $\psi\colon B \to B$ by $\psi(b) = \varphi(nb)$ for all $b \in B$. Since $R = End(B)$, there is some $r \in R$ such that $\psi(b) = rb$ for all $b \in B$. We infer that $r = r1 = \psi(1) = \varphi(n1) = n\varphi(1) \in nR$ and thus $r = ns$ for some $s \in R$. It follows that $(\varphi - s)(nx) = \varphi(nx) - nsx = \psi(x) - rx = 0$ for all $x \in R$ and $\varphi - s$ induces a homomorphism from the torsion group R/nR into the torsion-free group R^+. This shows that $\varphi = s \in R$, which shows that R is an E-ring. ∎

We have the following

PROPOSITION 2: *Let R be a torsion-free E-ring and $R \subseteq B \subseteq \mathbb{Q}R$ such that B is an R-module and B/R is bounded. Then $\alpha\colon \mathbb{Z} \to B$ where $\alpha(1) = 1 \in R$ is a quasi-localization of $\mathbb{Z}$.*

Proof: Let n be a natural number such that $nB \subseteq R \subseteq B$ and $\varphi \in Hom(\mathbb{Z}, B)$. Then there is some $b \in B$ such that $\varphi = b^*$ and $nb = r \in R$ with $nb^* = r \circ \alpha$. Since $\alpha(1) = 1 \in R$, the element $r \in R$ is unique. ∎

For example, let $R = \mathbb{Z}$, p a prime and $B = p^{-1}\mathbb{Z}$, then $\mathbb{Z} \subseteq B \subseteq \mathbb{Q}$ is a quasi-localization of $\mathbb{Z}$ but $B \neq \mathbb{Z}$. In the next section we will present a strategy that will be useful in the construction of quasi-localizations of $\mathbb{Z}$ with a given ring R.

3. E-forcing families of left ideals

The following result shows that E-forcing families need to be infinite in many cases.

PROPOSITION 3: *Let K be a field and A be a K-algebra without zero divisors and $A \neq K$. Then any E-forcing family $\mathcal{F}$ of ideals of A is infinite.*

Proof: Assume that $\mathcal{F}$ is a finite E-forcing family for A. Then $\prod_{X\in\mathcal{F}} X \subseteq \bigcap_{X\in\mathcal{F}} X = D \neq \{0\}$. Let $0 \neq d_1 \in D$ and $D = Kd_1 \oplus C_1$ and $A = D \oplus C_2$ as K-vector spaces. If $C_1 = \{0\}$, then $d_1^2 \in D$ is of the form $d_1^2 = d_1 k$ for some $0 \neq k \in K$. Since A has no zero divisors, we infer that $d_1 = k$ is a unit of A and therefore $D = A$. This implies that $\mathcal{F} = \{A\}$ is an E-forcing family for A and thus $End_K(A) = A$. But $End_K(A)$ has zero divisors if $\dim_K(A) > 1$. This shows that $A = K$ and we may assume that $C_1 \neq \{0\}$. Now write $C_1 = Kd_2 \oplus C_3$ with $d_2 \neq 0$ and define $\varphi, \psi \in End_K(A)$ by $\varphi(C_3 \oplus C_2) = \{0\} = \psi(C_3 \oplus C_2)$ and $\varphi(d_1) = d_1, \varphi(d_2) = 0$ and $\psi(d_1) = 0, \psi(d_2) = d_2$. Then $\varphi(A), \psi(A) \subseteq D \subseteq X$ for all $X \in \mathcal{F}$. On the other hand, $\varphi \circ \psi = 0$ and since A has no zero divisors, at least one of the two maps is not in A. This shows that $\mathcal{F}$ is not an E-forcing family for A. ∎

The next two propositions deal with inheriting E-forcing families.

PROPOSITION 4: *Let $1 \in R$ be a torsion-free ring such that the $\mathbb{Q}$-algebra $A = \mathbb{Q} \otimes_{\mathbb{Z}} R$ has an E-forcing family $\mathcal{F}$. Then $\mathcal{F}' = \{X \cap R : X \in \mathcal{F}\}$ is an E-forcing family for the ring R. (We identify R and $1 \otimes R$.)*

Proof: Let $X \in \mathcal{F}$ and $x \in X$. Then there exists some $n \in \mathbb{N}$ such that $nx \in X \cap R$. This shows that $X = \mathbb{Q} \otimes (X \cap R)$. Now let $\varphi \in End(R^+)$ such that $\varphi(X \cap R) \subseteq X \cap R$ for all $X \in \mathcal{F}$. Note that $\psi = id_{\mathbb{Q}} \otimes \varphi \in End_{\mathbb{Q}}(A)$ such that $\psi \restriction_R = \varphi$ and $\psi(X) \subseteq \mathbb{Q} \otimes \varphi(X \cap R) \subseteq X$. This shows that $\psi = a$ for some $a \in A$, but $\varphi(1) = \psi(1 \otimes 1) = a(1 \otimes 1) = a$ is in R and it follows that $\varphi \in R$. ∎

The next proposition shows that quasi-localizations of $\mathbb{Z}$ are induced by E-forcing families of its rings.

PROPOSITION 5: *Let $1 \in R$ be a torsion-free ring and $R \subseteq M \subseteq \mathbb{Q}R$ as R-modules such that $End_{\mathbb{Z}}(M) = R$. Then R has an E-forcing family $\mathcal{F}$.*

Proof: Let p be a prime and $n \in \mathbb{N}$. Let $(t_p(M/R))[p^n] = M_{p,n}/R$ and $X_{p,n} = p^n M_{p,n} \subset R$, a left ideal of R. Note that $p^n R \subseteq X_{p,n}$. Define

$\mathcal{F} = \{X_{n,p} : p$ prime, $n \in \mathbb{N}\}$. Let $\varphi \in End(R^+)$ and ψ the unique extension of φ to $\psi \in End(\mathbb{Q}R)$. Assume that $\varphi(X_{p,n}) \subseteq X_{p,n}$ for all p, n. Then $\psi(M_{p,n}) \subseteq M_{p,n}$ and since $M = \sum_{p,n} M_{p,n}$ we infer that $\psi(M) \subseteq M$, i.e. $\psi \restriction_M \in End(M) = R$. This shows that $\mathcal{F}$ is an E-forcing family of R. ∎

We have seen that E-forcing families of $\mathbb{Q}R$ induce E-forcing families of R. Sometimes it also works the other way around.

PROPOSITION 6: *Let $1 \in R$ be a ring such that R^+ is free of finite rank and $\mathcal{F}$ an E-forcing family of R such that each $X \in \mathcal{F}$ is pure in R. Then $\mathcal{F}' = \{\mathbb{Q}X : X \in \mathcal{F}\}$ is an E-forcing family for the $\mathbb{Q}$-algebra $\mathbb{Q}R$.*

Proof: Let ψ: $\mathbb{Q}R \to \mathbb{Q}R$ be a linear map such that $\psi(\mathbb{Q}X) \subseteq \mathbb{Q}X$ for all $X \in \mathcal{F}$. Since R^+ is finitely generated, there is some $m \in \mathbb{N}$ such that $m\psi(R) \subseteq R$. Thus $m\psi(X) \subseteq m\psi(\mathbb{Q}X \cap R) \subseteq \mathbb{Q}X \cap R = X$ since X is pure in R^+ for all $X \in \mathcal{F}$. This shows that $m\psi \restriction_R \in R$ and thus $\psi \in \mathbb{Q}R$. ∎

Now we consider matrix rings.

PROPOSITION 7: *Let S be a any ring, $1 \in S$, so that S has an E-forcing family $\mathcal{F}_S$ of left ideals of S. We may assume that $S \in \mathcal{F}_S$ and $R = Mat_{n\times n}(S)$ denotes the ring of $n \times n$-matrices over S. Then there exists an E-forcing family $\mathcal{F} = \{J_iX : 1 \leq i \leq n+1, X \in \mathcal{F}_S\}$ of left ideals of the ring R such that $R = \bigoplus_{i=1}^n J_i$ and $J_{n+1} \cap (\bigoplus_{1\leq j\neq i\leq n} J_i) = \{0\}$ for all $1 \leq j \leq n$. Moreover, if S is an E-ring, then R has an E-forcing family with $n+1$ members.*

Proof: Let $\varepsilon_{ij} \in R$ be the matrix with 1 in the (i,j)-position and 0 everywhere else. Let $\varepsilon_i = \varepsilon_{ii}$ and $J_i = R\varepsilon_i = \bigoplus_{1\leq\alpha\leq n} S\varepsilon_{\alpha i}$. Clearly $R = \bigoplus_{i=1}^n J_i$. Define $J_{n+1} = R\varepsilon^{(1)}$ where $\varepsilon^{(i)} = \sum_{j=1}^n \varepsilon_{ij}$. Then $J_{n+1} = \bigoplus_{i=1}^n S\varepsilon^{(i)}$, i.e. J_{n+1} is the set of all elements of R having constant rows. This shows that $J_{n+1} \cap (\bigoplus_{1\leq j\neq i\leq n} J_i) = \{0\}$ for all $1 \leq j \leq n$. If $r = (r_{\alpha\beta}) \in R$ then $r\varepsilon_{ij} = \sum_{\alpha=1}^n r_{\alpha i}\varepsilon_{\alpha j}$.

Let $\varphi \in End(R^+) = \bigoplus_{1\leq i,j,\alpha,\beta\leq n} Hom(S\varepsilon_{ij}, S\varepsilon_{\alpha,\beta})$ such that $\varphi(J_iX) \subseteq J_iX$ for all $1 \leq i \leq n+1$, and $X \in \mathcal{F}_S$. Then there are $\tau_{ij,\alpha\beta} \in End(S)$ such that $\varphi(x_{ij}\varepsilon_{ij}) = \sum_{1\leq\alpha,\beta\leq n} \tau_{ij,\alpha\beta}(x_{ij})\varepsilon_{\alpha\beta}$ for all $x_{ij} \in S$. Now $\varphi(J_i) \subseteq J_i$ implies that $\varphi(x_{ij}\varepsilon_{ij}) \in J_j = \bigoplus_{1\leq\gamma\leq n} S\varepsilon_{\gamma j}$ and thus $\tau_{ij,\alpha\beta} = 0$ for all $\beta \neq j$. We infer that $\varphi(x_i\varepsilon^{(i)}) = \varphi(\sum_{j=1}^n x_i\varepsilon_{ij}) = \sum_{j=1}^n \varphi(x_i\varepsilon_{ij}) = \sum_{j=1}^n(\sum_{\alpha=1}^n \tau_{ij,\alpha j}(x_i)\varepsilon_{\alpha j}) = \sum_{\alpha=1}^n(\sum_{j=1}^n \tau_{ij,\alpha j}(x_i))\varepsilon_{\alpha j} = \sum_{\alpha=1}^n c_{i\alpha}\varepsilon^{(\alpha)} = \sum_{\alpha=1}^n \sum_{j=1}^n c_{i\alpha}\varepsilon_{\alpha j}$ for some $c_{i\alpha} \in S$. This implies that $\tau_{ij,\alpha j}(x_i) = c_{i\alpha}$ for all $x_i \in S$, $1 \leq j \leq n$ and all $1 \leq i, \alpha \leq n$. This shows that $\tau_{ij,\alpha j} = \tau_{ik,\alpha k} =: \tau_{i\alpha}$ for all $1 \leq i, j, k \leq n$.

Moreover, since $\varphi(J_iX) \subseteq J_iX$ for all $1 \le i \le n+1$ and all $X \in \mathcal{F}_S$, we have that $\tau_{i\alpha} \in End(S^+)$ such that $\tau_{i\alpha}(X) \subseteq X$ for all $X \in \mathcal{F}_S$ and it follows that $\tau_{i\alpha} = t_{\alpha i} \in S$. Let $r = \sum_{j,i} x_{ij}\varepsilon_{ij} \in R$. Then

$$\begin{aligned}\varphi(r) &= \varphi\Big(\sum_{j,i} x_{ij}\varepsilon_{ij}\Big) = \sum_{j,i}\varphi(x_{ij}\varepsilon_{ij}) = \sum_{j,i}\Big(\sum_{1\le\alpha,\beta\le n}\tau_{ij,\alpha\beta}(x_{ij})\varepsilon_{\alpha\beta}\Big)\\ &= \sum_{j,i}\sum_{1\le\alpha\le n}\tau_{ij,\alpha j}(x_{ij})\varepsilon_{\alpha j} = \sum_{j,i,\alpha}\tau_{i\alpha}(x_{ij})\varepsilon_{\alpha j} = \sum_{j,i,\alpha} t_{\alpha i}x_{ij}\varepsilon_{\alpha j}\\ &= \sum_{j,\alpha}\Big(\sum_i t_{\alpha i}x_{ij}\Big)\varepsilon_{\alpha j} = tr,\end{aligned}$$

where $t = (t_{\alpha i}) \in R$. This shows that $\varphi \in R$. ∎

We continue to use the notation of the previous proposition. The above proof shows:

Corollary 1: *Let K be a $\mathbb{Q}$-algebra and $R = Mat_{n\times n}(K)$. Let $\varphi \in End_{\mathbb{Q}}(R)$ such that $\varphi(J_j) \subseteq J_j$ for all $1 \le j \le n+1$. Then there exists $\tau_{i\alpha} \in End_{\mathbb{Q}}(K)$ such that $\varphi(x_{ij}\varepsilon_{ij}) = \sum_{\alpha=1}^{n}\tau_{i\alpha}(x_{ij})\varepsilon_{\alpha j}$ for all $1 \le i,j \le n$.*

Corollary 2: *Let $1 \in S$ be a torsion-free ring and $K = \mathbb{Q}S$, a $\mathbb{Q}$-algebra. Let $V = K^{2m}$ be a free K-module of finite rank $2m$ and $\{V_i : 1 \le i \le 5\}$ be five K-submodules of V such that $\{\varphi \in End_{\mathbb{Q}}(V) : \varphi(V_i) \subseteq V_i$ for all $1 \le i \le 5\} = K$. Let $R = Mat_{(2m)\times(2m)}(S)$ and define*

$$\mathcal{F} = \{J_i : 1 \le i \le 2m+1\} \cup \{V_k^{\#} : 1 \le k \le 5\},$$

where $V_k^{\#}$ is the left ideal of R consisting of all matrices such that each row is an element of $V_k \cap S^{2m}$. Then $\mathcal{F}$ is an E-forcing family of left ideals of R.

Proof: Let $\varphi \in End(R^+)$ such that $\varphi(X) \subseteq X$ for all $X \in \mathcal{F}$. By the above, there are $\tau_{i\alpha} \in End(S^+)$ such that $\varphi(x_{ij}\varepsilon_{ij}) = \sum_{\alpha=1}^{2m}\tau_{i\alpha}(x_{ij})\varepsilon_{\alpha j}$ for all $1 \le i,j \le 2m$. Now suppose $\varphi(V_k^{\#}) \subseteq V_k^{\#}$ and $v_i = \sum_{\beta=1}^{2m} v_{i\beta}\varepsilon_{i\beta} \in V_k$. Then $\varphi(v_i) = \sum_{\beta=1}^{2m}\varphi(v_{i\beta}\varepsilon_{i\beta}) = \sum_{\beta,\alpha}\tau_{i\alpha}(v_{i\beta})\varepsilon_{\alpha\beta}$ and $\sum_\beta \tau_{i\alpha}(v_{i\beta})\varepsilon_{\alpha\beta} \in V_k$ for all $1 \le \alpha \le 2m$. This shows that $\tau_{i\alpha}^{\#} : S^{2m} \to S^{2m}$ defined by $\tau_{i\alpha}^{\#}(s_j)_j = (\tau_{i\alpha}(s_j))_j$ has the property that $\tau_{i\alpha}^{\#}(V_k \cap S^{2m}) \subseteq V_k \cap S^{2m}$ for all $1 \le k \le 5$ and $V_k = \mathbb{Q}(V_k \cap S^{2m})$. By our hypothesis, there is some $t_{\alpha_i} \in K$ such that $\tau_{i\alpha}^{\#} = t_{\alpha i} \in K$. Since $\tau_{i\alpha}^{\#}(S^{2m}) \subseteq S^{2m}$, it follows that $t_{\alpha i} \in S$. We now have that $\varphi(\sum_{ij} x_{ij}\varepsilon_{ij}) = \sum_{i,j,\alpha}\tau_{i\alpha}(x_{ij})\varepsilon_{\alpha j} = \sum_{i,j,\alpha} t_{\alpha i}x_{ij}\varepsilon_{\alpha_j} = (t_{\alpha i})_{\alpha,i}(x_{ij})_{i,j}$ and thus $\varphi \in R$. ∎

For future reference, we state a result due to Brenner [2], see also [1, Lemma 1] that shows that the hypothesis above holds for all torsion-free rings S of finite rank.

THEOREM 2 (See [2]): *Let K be a finite dimensional $\mathbb{Q}$-algebra generated by a set $\{1, \gamma_1, \gamma_2, \ldots, \gamma_{m-1}\}$. Then there exist five K-submodules V_k of K^{2m}, that are free summands of the K-module K^{2m}, such that*

$$\{\varphi \in End_{\mathbb{Q}}(K^{2m}) : \varphi(V_k) \subseteq V_k \text{ for all } 1 \leq k \leq 5\} = K.$$

Next we show that E-forcing families are inherited to quasi-equal rings.

PROPOSITION 8: *Let $1 \in R, S$ be torsion-free rings and $m \in \mathbb{N}$ such that $mR \subseteq S \subseteq R$ as rings.*

(1) *If $\mathcal{F}$ is an E-forcing family of left ideals of R, then $\mathcal{F}' = \{X \cap S : X \in \mathcal{F}\}$ is an E-forcing family of left ideals of S.*
(2) *If $\mathcal{F}$ is an E-forcing family of left ideals of S, then $\mathcal{F}' = \{RX : X \in \mathcal{F}\}$ is an E-forcing family of left ideals of R*

Proof: To show (1), let $\varphi \in End(S^+)$ such that $\varphi(X \cap S) \subseteq X \cap S$ for all $X \in \mathcal{F}$. Note that $mX \subseteq X \cap S$ and let $\psi = \varphi \restriction_{mR} \in Hom(mR, S)$. Define $\psi' \in End(R^+)$ by $\psi'(x) = \psi(mx)$ for all $x \in R$. Then $\psi'(X) = \psi(mX) = \varphi(mX) \subseteq \varphi(X \cap S) \subseteq X \cap S \subseteq X$. Since $\mathcal{F}$ is an E-forcing family for R, there is some $r \in R$ such that $\psi' = r$. This implies that $rx = \psi'(x) = \psi(mx) = \varphi(mx) = m\varphi(x)$ for all $x \in S$. For $x = 1 \in S$ it follows that $r = ms$ for $s = \varphi(1) \in S$. This shows that $\varphi = s$ is in S.

To show (2), let $\varphi \in End(R^+)$ and $X \in \mathcal{F}$. Then $X' = RX$ is a left ideal of R and $mX' = mRX \subseteq SX = X$. Assume $\varphi(X') \subseteq X'$ for all $X \in \mathcal{F}$. Then $m\varphi(X') \subseteq mX' \subseteq X$ and $m\varphi(X) \subseteq X$ for all $X \in \mathcal{F}$. This shows that $m\varphi \restriction_S = s$ for some $s \in S$ and $s = s1 = m\varphi(1) \in mR$ and thus $s = mr$ for some $r \in R$. For $x \in R$ we have $mx \in S$ and $m\varphi(x) = \varphi(mx) = mrx$. Since R^+ is torsion-free, we infer that $\varphi(x) = rx$ for all $x \in R$. ∎

Now we construct our first E-forcing family for polynomial rings. We already know by Proposition 3 that this family will be infinite.

LEMMA 1: *Let K be an infinite field. Then the K-algebra $K[x]$ of all polynomials in indeterminate x over K has an E-forcing family. Moreover, the ring $\mathbb{Z}[x]$ of integer polynomials has an E-forcing family $\mathcal{F}$ such that all members of $\mathcal{F}$ are direct summands of the free abelian group $(\mathbb{Z}[x])^+$.*

Proof: Let $\varepsilon_{n,j}$ be distinct elements of K for all $n \in \mathbb{N}$ and ordinals $1 \leq j < \omega$ and $\varepsilon_{n,0} = 0$ for all $n \in \mathbb{N}$. Let $\mathcal{F} = \{(\varepsilon_{n,j} + x^n)K[x] : n \in \mathbb{N}, j < \omega\}$. Let $\varphi \in End_K(K[x])$ such that $\varphi(X) \subseteq X$ for all $X \in \mathcal{F}$ and $\varphi(K) = \{0\}$. Then there exist polynomials $g_{n,j}$ such that $\varphi(\varepsilon_{n,j} + x^n) = (\varepsilon_{n,j} + x^n)g_{n,j}$ for all n and j. We infer that $(\varepsilon_{n,j} + x^n)g_{n,j} = x^n g_{n,0}$ for all n and $j \geq 1$. Since $\gcd(\varepsilon_{n,j} + x^n, x^n) = 1$ for all $j \geq 1$ we have that the polynomial $\varepsilon_{n,j} + x^n$ divides $g_{n,0}$ for all $j \geq 1$ and therefore $g_{n,0} = 0$ for all n. But this means that $\varphi = 0$. Now let $\psi \in End_K(K[x])$ such that $\psi(X) \subseteq X$ for all $X \in \mathcal{F}$. Then $\varphi = \psi - \psi(1)$ has the property that $\varphi(K) = \{0\}$ and the above shows that $\psi = \psi(1)$ as desired and $\mathcal{F}$ is an E-forcing family for the K-algebra $K[x]$. The case $K = \mathbb{Q}$ and Proposition 4 yield the desired result for $\mathbb{Z}[x]$. ∎

The following ring was introduced by A. L. S. Corner to obtain torsion-free abelian groups without indecomposable summands, cf. [9, page 145].

Let $\Lambda = \{\gamma : 0 \leq \gamma \in \mathbb{Q}\}$ and define a semigroup structure on Λ by setting $\alpha\beta = \max\{\alpha, \beta\}$ for all $\alpha, \beta \in \Lambda$.

Lemma 2: *Let Λ be the semigroup defined above and $R = S\Lambda$ the semigroup ring of Λ over the commutative ring S. Then*

$$\mathcal{F} = \{R\gamma : \gamma \in \Lambda\} \cup \{R(1-\gamma) : \gamma \in \Lambda\}$$

is an E-forcing family for the S-algebra R such that each member of $\mathcal{F}$ is a direct summand of the S-module R^+.

Proof: Let $\varphi \in End_S(R)$ such that $\varphi(S) = \{0\}$. Then there is a column-finite $\Lambda \times \Lambda$-matrix $M = [s_{\alpha,\beta}]_{\alpha,\beta\in\Lambda}$ such that $\varphi(\alpha) = \sum_{\beta\in\Lambda} \beta s_{\beta,\alpha}$ for all $\alpha \in \Lambda$. Now $R\gamma = \bigoplus_{\gamma\leq\alpha\in\Lambda} S\alpha$ is invariant for all $\gamma \in \Lambda$, which implies that $s_{\beta,\gamma} = 0$ for all $0 \leq \beta < \gamma$. Moreover, $s_{\beta,0} = 0$ for all $\beta \in \Lambda$. Note that $R(1-\gamma) = \langle \beta - \beta\gamma : \beta \in \Lambda \rangle = \langle \beta - \gamma : 0 \leq \beta < \gamma \rangle = \bigoplus_{0\leq\beta<\gamma} S(\beta-\gamma)$ is invariant under φ for all $0 < \gamma \in \Lambda$. This implies $\varphi(1-\gamma) = \varphi(-\gamma) = \sum_{\beta\geq\gamma} -\beta s_{\beta,\gamma} = \sum_{0\leq\beta<\gamma}(\beta - \gamma)t_{\beta,\gamma}$ for some $t_{\beta,\gamma} \in S$. This implies $-s_{\gamma,\gamma} = \sum_{0\leq\beta<\gamma} t_{\beta,\gamma}$ and $t_{\beta,\gamma} = 0$ for $0 \leq \beta < \gamma$. But this means that $s_{\gamma,\gamma} = 0$ for all $\gamma > 0$ as well as $s_{\beta,\alpha} = 0$ for all $\beta > \gamma$. This shows that M is the zero matrix and we have the desired result $\varphi = 0$. ∎

Now we consider rings of algebraic integers in algebraic number fields.

Theorem 3: *Let F be a Galois field extension of $\mathbb{Q}$ with finite Galois group G and S the ring of algebraic integers of F. Then S has an E-forcing family*

$\mathcal{F} = \{L_i : i < \omega\}$ *of prime ideals such that each* L_i *lies above a prime number* p_i *and* $(i \longmapsto p_i)$ *is one-to-one.*

Proof: First we define $\mathcal{F}$. Let $F = \mathbb{Q}[\pi]$ with minimal polynomial $m_\pi(x) \in \mathbb{Z}[x]$ of degree n. Then there exists an infinite set $\mathbb{P}$ of prime numbers such that $m_\pi(x)$ has a root $\bmod p$ for all $p \in \mathbb{P}$ (see [3, Proposition on page 298]) and we may assume that p is not ramified for any prime $p \in \mathbb{P}$. Thus, for $p \in \mathbb{P}$, we have that one prime ideal lying over p has dimension 1 $\bmod p$, but G operates transitively on those prime ideals and only id_F fixes any of them because there are n such ideals. Now define $\mathcal{F} = \{P_p : p \in P\}$ where the prime ideal P lies above p. Let $\{a_1, a_2, \dots, a_n\}$ be an integral basis of S and $G = \{g_1, g_2, \dots, g_n\}$ with $g_1 = id_F$. Since G is linearly independent over F, we have $\mathbb{Q}(SG) = \mathbb{Q}(End(S^+)) = End_{\mathbb{Q}}(F)$, where $SG = \{\sum_{i=1}^n s_i g_i : s_i \in S\}$. Define an $n \times n$-matrix Δ over S by $\Delta = [g_i(a_j)]_{1 \le i,j \le n}$. Then $\det(\Delta) \ne 0$ and there exists a (least) number $m_\Delta \in \mathbb{N}$ such that $m_\Delta \Delta^{-1} \in Mat_{n\times n}(S)$. Let $\varphi \in End(S^+)$ such that $\varphi(P) \subseteq P$ for all $P \in \mathcal{F}$. Then there exists some $m \in \mathbb{N}$ such that $m\varphi = \sum_{i=1}^n s_i g_i \in SG$. Mader and Vinsonhaler [12] used the following trick in the proof of their Lemma 2.5.

Let $f = m\varphi$ and note that $f(P) \subseteq P$ for all $P \in \mathcal{F}$. Observe that

$$(f(a_1), \dots, f(a_n)) = \Big(\sum_{i=1}^n s_i g_i(a_1), \dots, \sum_{i=1}^n s_i g_i(a_n)\Big) = (s_1, s_2, \dots, s_n)\Delta.$$

Now let $x \in P \in \mathcal{F}$. Then

$$\begin{aligned}(f(xa_1), \dots, f(xa_n)) &= (\sum_{i=1}^n s_i g_i(xa_1), \dots, \sum_{i=1}^n s_i g_i(xa_n)) \\ &= (\sum_{i=1}^n s_i g_i(x) g_i(a_1), \dots, \sum_{i=1}^n s_i g_i(x) g_i(a_n)) \\ &= (s_1 g_1(x), \dots, s_n g_n(x))\Delta \in P \times P \times \cdots \times P.\end{aligned}$$

Note that $m_\Delta \Delta^{-1}$ is a matrix with entries in S thus

$$(f(xa_1), \dots, f(xa_n)) m_\Delta \Delta^{-1} = m_\Delta (s_1 g_1(x), \dots, s_n g_n(x)) \in P \times P \times \cdots \times P$$

and we infer that $m_\Delta s_i g_i(P) \subseteq P$ for all $1 \le i \le n$. Define

$$\Pi_i = \{p \text{ prime} : \exists P \in \mathcal{F} \text{ such that } pS \subseteq P,\ g_i(P) \ne P \text{ and } \gcd(p, m_\Delta) = 1\}.$$

By our hypotheses, Π_i is infinite for all $2 \le i \le n$. If $p \in \Pi_i$ and $pS \subseteq P \in \mathcal{F}$, then S/P is a torsion p-group and $m_\Delta s_i g_i(P) \subseteq P$ implies that $s_i g_i(P) \subseteq P$

and $g_i(P) \neq P$. Since $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we have that P is a maximal ideal of S, therefore, $S = g_i(P) + P$ and we have $1 = g_i(a) + b$ for some elements $a, b \in P$. This implies that $s_i = s_i g_i(a) + s_i b \in P + P = P$ for all $2 \leq i \leq n$ and $p \in \Pi_i$. Since Π_i is infinite, we have that each s_i belongs to infinitely many P's and we infer $s_i = 0$ for all $2 \leq i \leq n$, thus $f = m\varphi = s_1 id_F$ and $m\varphi(1) = s_1 \in mS$. Since S^+ is torsion-free, we may cancel m and get $\varphi \in S$. ∎

COROLLARY 3: *Let S be the ring of algebraic integers of the quadratic number field $F = \mathbb{Q}[\sqrt{m}]$. Then S has an E-forcing family of prime ideals. So does $\mathbb{Z}[\sqrt{m}]$.*

Proof: Let $G = \{id_F, \sigma\}$ be the Galois group of F. Assume that $m > 0$. By Dirichlet's Arithmetic Progression Theorem, the set $\Pi = \{p \text{ prime} : p \equiv 1 \bmod 4m\}$ is *infinite*. Obviously, for $p \in \Pi$, $p \equiv 1 \bmod m$ is a quadratic residue $\bmod\, m$ and $p \equiv 1 \bmod 4$. By Gauss' Quadratic Reciprocity Theorem, we have that m is a quadratic residue $\bmod\, p$ for all $p \in \Pi$. Since $p \equiv 1 \bmod 4$ for each $p \in \Pi$, we have that -1 is a quadratic residue $\bmod\, p$ as well. This shows that any $m \in \mathbb{Z}$ is a quadratic residue $\bmod\, p$ for all $p \in \Pi$. Let Π' be the co-finite subset of Π of all primes $p \in \Pi$ such that p is unramified in S. Then $pS = P_p Q_p$ with distinct prime ideals P_p, Q_p of S. Since G operates transitively on the set $\{P_p, Q_p\}$, we have that $\sigma(P_p) = Q_p$ for all $p \in \Pi'$. The family $\mathcal{F} = \{P_p : r \in \Pi'\}$ now has the required properties to apply the above theorem. ∎

Example 1: (Compare with Example in [14, page 987].) Consider $S = \mathbb{Z}[i]$ with $i^2 = -1$. Let Σ be a nonempty set of primes p such that $p \equiv 3 \bmod 4$ for all $p \in \Sigma$. Let $R = S_\Sigma$ be the localization of S at the set Σ of primes. If $0 \neq a + ib \in \mathbb{Z}[i]$ then $(a+ib)^{-1} = \frac{a-ib}{a^2+b^2} \in \mathbb{Q}[i]$ and $a^2 + b^2 \neq 0 \bmod p$ for all $p \in \Sigma$ unless $a + ib \in p\mathbb{Z}[i]$. This shows that each ideal J of R has the form $J = qR$ for some $q \in \mathbb{N}$. This shows that the ring R has no E-forcing family of (left) ideals since $\varphi(J) \subseteq J$ for all $\varphi \in End(R^+)$, which is a 2×2-matrix ring and thus not isomorphic to R. We infer that R is not the ring of a quasi-localization of $\mathbb{Z}$.

4. From E-forcing families to modules

Recall that if A is a torsion-free abelian group and τ is a type, then $A(\tau) = \{a \in A : a \text{ has type} \geq \tau\}$. If G is any abelian group then $t_p(G)$ denotes the p-primary part of the torsion subgroup $t(G)$ of G. First we need

Lemma 3: *Let G be a torsion-free abelian group such that G is homogeneous of type 0. Let $\{V_i : i \in I\}$ be a family of at most countably many pure subgroups of G such that:*

(1) $G = \sum_{i\in I} V_i$ *and*

(2) *Each G/V_i is homogeneous of type* 0.

Let $\{P_i : i \in I\}$ be a family of disjoint infinite sets of primes and $R_i = \langle p^{-1} : p \in P_i\rangle \subset \mathbb{Q}$. Let τ_i denote the type of R_i for all $i \in I$ and define $M = \sum_{i\in I} R_i V_i$. Then $M(\tau_i) = (V_i)_*$ is the purification of V_i in M.

Proof: Recall that if A is any abelian group, $n, m \in \mathbb{N}$ such that $\gcd(m, n) = 1$, then $ma \in nA$ for some $a \in A$ implies $a \in nA$. By definition, $V_i \subseteq M(\tau_i)$ and $M(\tau_i)$ is a pure subgroup of M, which implies that $(V_i)_* \subseteq M(\tau_i)$. To show the other inclusion, fix $i \in I$ and let $s \in M(\tau_i)$. Since M/G is torsion, there is some $m \in \mathbb{N}$ such that $s' = ms \in G \cap M(\tau_i)$. Note that

$$P_i' = \{p \in P_i : s' \in pM, s' \notin pG, \gcd(p, m) = 1\}$$

is co-finite in P_i. Let Π_i be the set of all square-free natural numbers whose prime factors are contained in P_i. Let $p \in P_i'$. Then $px = s'$ for some $x \in M$ such that $x = \sum_j (1/q_j)v_j$ for some $v_j \in V_j$ and $q_j \in \Pi_j$. Let $q = \prod_{j\neq i} q_j$ and note that $\gcd(p, q) = 1$. Now $qx = q\sum_j (1/q_j)v_j = \sum_j (q/q_j)v_j = g + (1/q_i)v_i'$ with $v_i' = qv_i$. This implies that $qs' = pqx = pg + (p/q_i)v_i'$ and $pv_i' \in q_i G \cap V_i = q_i V_i$.

Assume p does not divide q_i. Then $v_i' \in q_i V_i$ and $v_i' = q_i v_i''$ for some $v_i'' \in V_i$ and thus $qs' = p(g + v_i'')$. Since $p \in P_i'$ we have $\gcd(p, q) = 1$ and thus $s' \in pG$, a contradiction to the definition of P_i'.

Thus we may assume that $q_i = pt$ for some $t \in \mathbb{N}$ and $\gcd(p, t) = 1$ since q_i is square-free. This implies that $qts' = ptg + (pt/q_i)v_i' = ptg + v_i'$ and we have that $qt(s' + V_i) \in p(G/V_i)$ and $\gcd(p, qt) = 1$ implies that $s' + V_i \in p(G/V_i)$ for all $p \in P_i'$. This shows that $\operatorname{type}(s' + V_i) \geq \tau_i > 0$ but G/V_i is homogeneous of type 0, which means that $s' + V_i = V_i$ and thus $s' \in V_i$. It follows that $s \in (V_i)_*$ since $ms = s' \in V_i$. ∎

Corollary 4: *Let $R = \mathbb{Z}[x]$ or $R = \mathbb{Z}\Lambda$ be the ring defined in Lemma 2. Then there exists $R \subseteq M \subseteq \mathbb{Q}R$ such that $End_{\mathbb{Z}}(M) = R$ and $p(t_p(M/R)) = 0$ for all prime numbers p.*

Proof: We have seen in Lemma 1 and Lemma 2 that both rings have countable E-forcing family $\mathcal{F} = \{V_i : i \in I\}$ such that each V_i is a direct summand of R^+ and $R = \sum_{i\in I} V_i$. Let $R_i \subseteq \mathbb{Q}$ be as described in Lemma 3 and set

$M = \sum_{i\in I} R_i V_i \subseteq \mathbb{Q}R$. Let $\varphi \in End_{\mathbb{Z}}(M)$ such that $\varphi(1) = 0$. By Lemma 3, $\varphi(V_i) \subseteq \varphi((V_i)_*) \subseteq \varphi(M(\tau_i)) \subseteq M(\tau_i) = (V_i)_*$, the purification of V_i in M. By Proposition 6 (1), $\mathcal{F}' = \{\mathbb{Q}V_i : i \in I\}$ is an E-forcing family of the $\mathbb{Q}$-algebra $\mathbb{Q}R$. Let $\psi \in End(\mathbb{Q}R)$ be the unique extension of φ. Then $\psi = r/q$ for some $r \in R$, $q \in \mathbb{N}$ since $\mathcal{F}'$ is E-forcing and $\psi(1) = \varphi(1) = 0$. This implies that $\psi = \varphi = 0$. Now let $\gamma \in End(M)$. Then there is some $m = \gamma(1) \in M$ and $(\gamma - m)(1) = 0$ implies, by the above, $\gamma = m$. We need to show that $m \in R$. To this end, note that all the orders $o(x)$ of the elements in $x \in M/R$ are square free. Let $m = s/q$ with $s \in R$ and $q = o(m + R)$. Then $\varphi(m) = m^2 = (s^2/q^2) \in M$. Let p be a prime divisor of q and $q = pq'$. Then $\gcd(p, q') = 1$ and $(q')^2(s^2/q^2) = s^2/p^2 \in M$. Thus $o((s^2/p^2)+R)$ divides p^2 and is square-free, i.e. $p(s^2/p^2) = s^2/p \in R$. Thus $s^2 \in pR$ and it follows that $(s + pR)^2 = 0 \in R/pR$. It is easy to see that in both cases of R, this implies that $s \in pR$, a contradiction to the choice of q and p. This shows that $q = 1$ and thus $\varphi = s \in R$. ∎

The following result will be used to construct quasi-localizations of $\mathbb{Z}$ whose rings are rings of algebraic integers.

PROPOSITION 9: *Let $1 \in S$ be a ring such that S^+ is a free abelian group of finite rank and $\mathcal{F} = \{P_i : i < \omega\}$ is an E-forcing family of right ideals of S such that:*

For each $i < \omega$ there is a (unique) number p_i such that $p_i S \subsetneqq P_i$ and the ring $S_i = P_i/p_i S$ has the property that $x \in S_i$ with $x^2 = 0$ implies $x = 0$.

Then there exists a right S-module M such that $S \subseteq M \subseteq \mathbb{Q}S$ and $End_{\mathbb{Z}} M = S$. Moreover, $p(t_p(M/S)) = 0$ for all prime numbers p.

Proof: Define $M = \sum_{i<\omega} p_i^{-1} P_i \subseteq \mathbb{Q}S$ and let $\varphi \in End(M)$. Since S^+ is finitely generated and M/S is torsion, there is some $k \in \mathbb{N}$ such that $k\varphi \restriction_S = \psi \in End(S^+)$. Since $t_{p_i}(M/S) = (p_i^{-1} P_i)/S$ we have that $k\varphi(p_i^{-1} P_i) \subseteq p_i^{-1} P_i$ and it follows that $\psi(P_i) \subseteq P_i$ for all $i < \omega$. Since $\mathcal{F}$ is an E-forcing family, we infer that $\psi = s \in S$ and $\varphi = s/k \in End(M)$. Note that $\varphi(1) = s/k \in M$. By definition of M, there exists a finite subset I of ω, elements $u \in S$ and $b_i \in P_i - p_i S$ such that $s/k = \sum_{i\in I} b_i/p_i + u$. Fix $j \in I$ and define $q = \prod_{i\in I-\{j\}} p_i$. Then $q\frac{s}{k} = \frac{qb_j}{p_j} + w$ for some $w \in S$ and still $q(s/k) \in End(M)$. This implies $\frac{b_j}{p_j}\frac{qs}{p_j} = \frac{qb_j^2}{p_j^2} + \frac{b_j w}{p_j} \in M$ and we infer that $\frac{qb_j^2}{p_j} \in S$ since all elements in M/S have square-free orders. This means that $q(b_j + p_j S)^2 = 0 \in S_i$ and $\gcd(q, p_j) = 1$. By our hypothesis, this implies that $b_j \in p_j S$, a contradiction to the choice of b_j, which shows $I = \emptyset$ and thus $\varphi = s/k \in S$, as desired. ∎

Corollary 5: *Let R be the ring of algebraic integers of a quadratic number field. Then there is an $R \subseteq M \subseteq \mathbb{Q}R$ such that $End(M) = R$. The same holds if R is the ring of algebraic integers of some algebraic number field F satisfying the hypotheses of Theorem 2.*

We will need the following

Lemma 4: *Let $1 \in R$ be a torsion-free ring and $\mathcal{F} = \{L_i : i < \omega\}$ a countable family of left ideals such that $L_i = Rb_i$, where b_i is not a zero-divisor in R and for each $i < \omega$ there is a prime number p_i and $\gamma_i \geq 1$ such that $p^{\gamma_i}\delta_i R \subseteq L_i$ where $\gcd(p_i, \delta_i) = 1$ and $(i \longmapsto p_i)$ is one-to-one. Let $M = R + \sum_{i<\omega} p^{-\gamma_i} L_i \subseteq \mathbb{Q}R$. If $y \in M$ and $y \in End(M^+)$, then $y \in R$.*

Proof: Note that $t_{p_i}(M/R) = (p_i^{-\gamma_i} L_i + R)/R$ and $t_p(M/R) = 0$ for $p \notin \{p_i : i < \omega\}$. Since M/R is torsion, $y = v/k$ with $v \in R$ and k is the order of $y + R$ in M/R. We may assume that $k = p_i$ for some $i < \omega$. Then $\left(\frac{v}{p_i} p_i^{-\gamma_i} L_i + R\right)/R \in t_{p_i}(M/R) = (p_i^{-\gamma_i} L_i + R)/R$. This implies that $p_i^{-(\gamma_i+1)} v L_i \subseteq p_i^{-\gamma_i} L_i + R$ and therefore $p_i^{-1}\delta_i v L_i \subseteq \delta_i L_i + p_i^{\gamma_i}\delta_i R \subseteq L_i + L_i = L_i$. This shows that $v\delta_i L_i \subseteq p_i L_i$ and $\gcd(\delta_i, p_i) = 1$ implies that $v L_i \subseteq p_i L_i$. We infer that there is some $r \in R$ such that $vb_i = p_i r b_i$. Since b_i is not a zero-divisor in R, we get that $v = p_i r$ and thus $k = 1$ and $y \in R$. ∎

Because of the relevance to our topic, we want to give a proof of Zassenhaus' result [18] that uses some ideas of Butler's [3]. We deem our version to be a little more elementary than the originals.

We begin with

Lemma 5: *Let F be a free abelian group of finite rank, $0 \neq e \in F$ and $\tau \in End(F)$. Let $W = e\mathbb{Z}[\tau]$ be the τ-invariant subgroup of F generated by e and W_* the purification of W in F. Then there exists (a least) $k \in \mathbb{N}$ such that $kW_* \subseteq W$. Let $c \in \mathbb{Z}$ such that c is not an eigenvalue of τ and $\alpha \in \mathbb{N}$. Then $\alpha e \in F(c - \tau)$ implies that $\det(c - \tau \restriction_W)$ divides $k\alpha$.*

Proof: Let $\chi_\tau(x) = \det(x - \tau)$ be the characteristic polynomial of τ. Then $\chi_\tau(x) \in \mathbb{Z}[x]$ and is monic. Thus $m_\tau(x)$, the minimal polynomial of τ, is in $\mathbb{Z}[x]$ and is monic as well. Let $f(x)$ be the minimal polynomial of $\tau \restriction_W$. Then $f(x)$ divides $m_\tau(x)$ and $f(x) = x^m + \sum_{i=0}^{m-1} a_i x^i \in \mathbb{Z}[x]$. We infer that $W = \bigoplus_{i=1}^{m-1} e\tau^i\mathbb{Z}$. Note that $F = W_* \oplus C$ and thus $\mathbb{Q}F = \mathbb{Q}W \oplus \mathbb{Q}C$ and since c is not an eigenvalue of τ, we have that $c - \tau$ is the root of an integer polynomial with nonzero constant term. Thus $(c-\tau)^{-1} \in \mu^{-1}\mathbb{Z}[c-\tau]$ for some $\mu \in \mathbb{N}$, but any polynomial in $c-\tau$ is also a polynomial in τ and thus $(c-\tau)^{-1} \in \mu^{-1}\mathbb{Z}[\tau]$.

Assume that $\alpha e \in F(c-\tau)$. Then $(\alpha e)(c-\tau)^{-1} \in \mu^{-1}W_* \cap F = W_*$, since W_* is pure in F. This shows that $\alpha e \in W_*(c-\tau)$ and thus $k\alpha e \in W(c-\tau) = (\bigoplus_{i=0}^{m-1} e\tau^i\mathbb{Z})(c-\tau)$. Define the $m \times m$-matrix $C(f) = (u_{ij})_{1\le i,j\le m}$ where

$$u_{ij} = \begin{cases} [c]c1 & \text{if } i = j+1,\ 1 \le j \le m-1 \\ -a_{i-1} & \text{if } j = m \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $C(f)$ is known as the companion matrix of the monic polynomial $f(x)$. Define $B = cI_{m\times m} - C(f)$. Suppose $k\alpha e = (\sum_{i=0}^{m-1} e\tau^i z_i)(c-\tau)$ and

$$\overrightarrow{z} = \begin{bmatrix} [c]cz_0 \\ \vdots \\ z_{m-1} \end{bmatrix} \in \mathbb{Z}^m.$$

Elementary computations show that we get

$$B\overrightarrow{z} = \begin{bmatrix} [c]ck\alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $\chi_{C(f)}(x) = \det(xI_{m\times m} - C(f)) = f(x)$. Multiplying from the left by the adjugate $adj(B)$ of B, we get that $\det(B)z_{m-1} = c_{1m}k\alpha$ where c_{1m} is the $(1,m)$-cofactor of the matrix B. It is easy to see that $c_{1m} = (-1)^{m-1}$. This proves that $f(c) = \det(B)$ divides $k\alpha$. ∎

We are now ready to prove Zassenhaus' result;

Theorem 4 (Zassenhaus [18]): *Let $1 \in R$ be a ring such that R^+ is free abelian of finite rank. Then there exists a left R-module M such that $R \subseteq M \subseteq \mathbb{Q}R$ and $End(M^+) = R$, i.e., all additive endomorphisms of M are left-multiplications by elements of R. Moreover, $t_p(M/R)$ is bounded for all primes p.*

Proof: Let $\Sigma = \{\sigma \in End(R^+) : 0 \ne \sigma \text{ and } \sigma(1) = 0\} = \{\sigma_i : i < \omega\}$. For each $i < \omega$, there is some $\tau_i \in R$ such that $\sigma_i(-\tau_i) = e_i \ne 0$. Using the notation as in the above lemma, for any $c_i \in \mathbb{Z}$ such that c_i is not an eigenvalue of τ_i, i.e. $c_i - \tau_i$ is a unit in the ring $End(\mathbb{Q}R^+)$, and there is some k_i such that $\alpha e \in R(c_i - \tau_i)$ implies that $\det(c_i - \tau_i \restriction_W) = f_i(c_i)$ divides αk_i. There are infinitely many primes q such that $f_i(x) \bmod q$ has a root in $\mathbb{Z}/q\mathbb{Z}$. This is a well-known result in number theory, an elementary proof of which is contained in [3]. Now pick such a prime $q \notin \{p_j : 0 \leq j \leq i-1\}$ that does not divide k_i. The set $\{c \in \mathbb{Z} : f_i(c) \equiv 0 \bmod q\}$ is infinite since any integer of the form $c^{(t)} = c+tq$, where $t \in \mathbb{Z}$ is a root of $f_i(x) \bmod q$ and we may choose c_i such that c_i is not one (of the finitely many) eigenvalues of τ and $f_i(c_i) \equiv 0 \bmod q$. Then $\det(c_i - \tau) = q^{\gamma_i}\delta_i$ and q does not divide δ_i. This implies that there is some $\rho \in R$ such that $\rho(c_i - \tau_i) = q^{\gamma_i}\delta_i$ and it follows that $q^{\gamma_i}\delta_i R \subseteq R(c_i - \tau_i)$. Moreover, $\delta_i\sigma_i(c_i - \tau_i) = \delta_i\sigma_i(-\tau_i) \notin R(c_i - \tau_i)$ since q divides $f_i(c_i) = \det(c_i - \tau \restriction_W)$, but q does not divide $\delta_i k_i$. Now set $L_i = R(c_i - \tau_i)$ and $p_i = q$. Let $M = R + \sum_{i<\omega} p_i^{-\gamma_i} L_i \subseteq \mathbb{Q}R$. Let $\varphi \in End(M^+)$. Since R^+ is finitely generated and M/R is torsion, there is some $m \in \mathbb{N}$ such that $m\varphi(R) \subseteq R$ and $m\varphi(1) \in R$. Let $\sigma = m\varphi - m\varphi(1) \in End(R^+)$, note that $\sigma(1) = 0$. Assume that $0 \neq \sigma$, then $\sigma = \sigma_i$ for some $i < \omega$ and thus $\delta_i\sigma_i(c_i - \tau_i) \notin L_i$. On the other hand, σ_i induces an endomorphism of M/R and it follows that $\sigma_i(p_i^{-\gamma_i}L_i) \subseteq p_i^{-\gamma_i}L_i + R$ and we have that $\delta_i\sigma_i(L_i) \subseteq \delta_i L_i + p_i^{\gamma_i}\delta_i R \subseteq L_i + L_i = L_i$. This contradiction shows that $\sigma_i = 0$ and therefore $\varphi = \varphi(1) \in End(M^+)$ and $\varphi(1) \in M$. Now Lemma 4 implies $\varphi(1) \in R$, and we are done. ∎

Now we deal with matrix rings.

Theorem 5: *Let $1 \in S$ be a torsion-free ring of finite rank and $K = \mathbb{Q}S$ generated by a set $\{1, \gamma_1, \ldots, \gamma_{m-1}\}$ as a $\mathbb{Q}$-algebra. Assume that there are $2m+6$ distinct primes p_i such that S^+ is p_i-reduced for all $1 \leq i \leq 2m+6$. Then $R = Mat_{(2m)\times(2m)}(S)$ is the ring of a quasi-localization M of $\mathbb{Z}$. Moreover, M/R is a direct sum of p_i-torsion divisible groups.*

Proof: Let $\mathcal{F}$ be the family given in Corollary 1. Let $L_i = J_i\mathbb{Z}[1/p_i]$ for $1 \leq i \leq 2m+1 = n+1$ as defined in Theorem 2, and $L_{2m+1+k} = V_k^{\#}\mathbb{Z}[1/(p_{2m+1+k})]$ for $1 \leq k \leq 5$. Define $M = R + \sum_{j=1}^{2m+6} L_i$. Let τ_i be the type of $\mathbb{Z}[1/p_i]$. An argument similar to [1, Theorem 1] shows that $M(\tau_i) = L_i$ for all $1 \leq i \leq 2m+6$. Let $\varphi \in End(M)$. Then $\varphi(M(\tau_i)) \subseteq M(\tau_i)$ and if ψ is the extension of φ to $\mathbb{Q}R$ then $\psi(\mathbb{Q}L_i) \subseteq \mathbb{Q}L_i$ and by our hypothesis there are $t_{\alpha i} \in K$ such that $\psi(x_{ij}\varepsilon_{ij}) = \sum_\alpha \tau_{i\alpha}(x_{ij})\varepsilon_{\alpha j} = \sum_\alpha t_{\alpha i}x_{ij}\varepsilon_{\alpha j}$. It follows that $\psi = t \in K$ where $t = (t_{\alpha i})$. The fact that $tL_i \subseteq L_i$ for all $1 \leq i \leq 2m+1$ implies that $t \in R$. ∎

Corollary 6: *Let $n \geq 2$ and $R = Mat_{n\times n}(E)$ be the ring of $n \times n$-matrices over E where E is some E-ring such that E^+ is p_i-reduced for distinct prime numbers $p_i, 1 \leq i \leq n+1$. Then there exists a quasi-localization M of $\mathbb{Z}$ whose ring is R such that $M/R \approx \bigoplus_{i=1}^{n+1} \mathbb{Z}(p_i^\infty)$.*

Corollary 7: *Let $n \geq 2$ and $R = Mat_{n\times n}(\mathbb{Z})$ be the ring of $n \times n$ integer matrices and $p_i, 1 \leq i \leq n+1$, be distinct prime numbers. Then there exists a quasi-localization M of $\mathbb{Z}$ whose ring is R such that $M/R \approx \bigoplus_{i=1}^{n+1} \mathbb{Z}(p_i^\infty)$. Moreover, M is a finite rank Butler group.*

5. Large Polynomial Rings

In this section we will show that commutative polynomial rings with uncountably many variables are rings of quasi-localizations of $\mathbb{Z}$. We will employ a Black Box construction very similar to the construction of E(R)-algebras in [10, Section 2]. The combinatorics will be the same, only the "Step Lemma" will be a little different. The case of polynomial rings in countably many variables requires a different method of construction and will be addressed in another paper. We fix the following

Notation 1: Let S be a commutative ring, $1 \in S$, such that S^+ is cotorsion-free, i.e. $Hom(\widehat{\mathbb{Z}}, S^+) = 0$, where $\widehat{\mathbb{Z}}$ is the $\mathbb{Z}$-adic completion of $\mathbb{Z}$. Let κ, μ, λ be infinite cardinals such that $\kappa \geq |S|$, $\mu^\kappa = \mu$ and $\lambda = \mu^+$ is the successor cardinal of μ.

Let $B = S[x_\alpha : \alpha < \lambda]$ be the ring of polynomials in the commuting variables x_α, $\alpha < \lambda$, and $\mathfrak{M}$ the set of all monomials, $1 \in \mathfrak{M}$. Then $B = \bigoplus_{m\in\mathfrak{M}} Sm$. For any $g = (g_m m)_{m\in\mathfrak{M}} \in \widehat{B} \subseteq \Pi_{m\in\mathfrak{M}}\widehat{S}m$, the **support** of g is defined as $[g] = \{m \in \mathfrak{M} : g_m \neq 0\}$. If $M \subseteq \widehat{B}$, then the support of M is $[M] = \bigcup_{g\in M}[g]$. Note that $[g]$ is at most countable for all $g \in \widehat{B}$. Define the X-support of g by $[g]_X = \{\alpha < \lambda : x_\alpha \text{ occurs in some } m \in [g]\} \subset \lambda$.

As usual, a norm is defined by $\|\{\alpha\}\| = \alpha + 1$ for $\alpha < \lambda$, and $\|M\| = \sup\{\|\{\alpha\}\| : \alpha \in M\}$ for any $M \subseteq \lambda$. Moreover, $\|g\| = \min\{\beta < \lambda : [g]_X \subseteq \beta\}$ for any $g \in \widehat{B}$. Note that $[g]_X \subseteq \beta$ holds if and only if $g \in \widehat{B_\beta}$ where $B_\beta = S[x_\alpha : \alpha < \beta]$.

Canonical homomorphisms are defined as in [10, Definition 2.1.1]. All we need to know here is that if φ is a canonical homomorphism then $\varphi\colon P \to \widehat{B}$ such that $P = S[x_\alpha : \alpha \in I]$ for some $I \subseteq \lambda$ with $|I| \leq \kappa$ and $\varphi(P) \subseteq \widehat{P}$. We define $[\varphi] = [P]$, $[\varphi]_X = [P]_X = I$ and $\|\varphi\| = \|P\| = \sup\{\alpha : \alpha \in I\}$.

We now have the Black Box as in [10]:

Theorem 6 ([10]): *Let E be a stationary subset of λ consisting only of ordinals of countable cofinality such that $\lambda - E$ is stationary as well. Given our notation as stated above. There exists a family $\{\varphi_\beta : \beta < \lambda\}$ of canonical homomorphisms such that*

(i) $\|\varphi_\beta\| \in E$ *for all* $\beta < \lambda$;
(ii) $\|\varphi_\gamma\| \leq \|\varphi_\beta\|$ *for all* $\gamma \leq \beta < \lambda$;
(iii) $\|[\varphi_\gamma]_X \cap [\varphi_\beta]_X\| < \|\varphi_\beta\| (= \|[\varphi_\beta]_X\|)$ *for all* $\gamma, \beta < \lambda$.
(iv) PREDICTION: *For any homomorphism* $\psi: B \to \widehat{B}$ *and for any subset* I *of* λ *with* $|I| \leq \kappa$ *the set*

$$\{\alpha \in E : \exists \beta < \lambda \text{ such that } \|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi, I \subseteq [\varphi_\beta]_X\} \text{ is stationary in } \lambda.$$

We are now ready for our Step Lemma.

LEMMA 6: *Let* $P = S[x_\alpha : \alpha \in I^*]$ *for some* $I^* \subseteq \lambda$ *and* M *a subgroup such that* $P \subseteq M \subseteq \widehat{P}$ *and* M_*, *the purification of* M *in* $\widehat{S}$, *is cotorsion-free. Moreover, assume that there is some subset* Y *of* $\widehat{P}$ *such that* Y *is algebraically independent over the ring* P, *i.e., the ring generated by* Y *and* P *is* $R := P[Y]$, *a polynomial ring. We assume that* M *is an* R*-module and* $R \subseteq M$ *such that* M/R *is torsion. Suppose there is a subset* $I = \{\alpha_n : n < \omega\}$ *of* I^* *with* $\alpha_n < \alpha_{n+1}$ *for all* $n < \omega$, *such that* $I \cap [g]_X$ *is finite for all* $g \in M$. *Let* $\varphi: P \to M$. *Then there exists an element* $y \in \widehat{P}$ *such that*

(1) *y is algebraically independent over R;*
(2) *there exists an $R[y]$-submodule M' of $\widehat{P}$ such that $R[y] \subset M$ and M is pure in M';*
(3) *either there is some $n \in \mathbb{N}$ such that $n\varphi \in R$ or $\varphi(y) \notin M'$;*
(4) *$(M')_*$ the purification of M' in $\widehat{S}$ is cotorsion-free and $M'/R[y]$ is torsion;*
(5) *if $n\varphi \in R$ for some $n \in \mathbb{N}$, then $\varphi \in R$.*

(The element y can be chosen to be $x := \sum_{n<\omega}(n!)x_{\alpha_n}$ or $y = x + \pi b$ for some element $\pi \in \widehat{\mathbb{Z}}$ and $b \in P$.)

Remark 1: We have to be a bit more careful than in the corresponding proof in [10], because our M and M' are not pure in $\widehat{P}$.

Proof: Let $y = \sum_{n<\omega}(n!)x_{\alpha_n}$. Then $[y]_X \cap [g]_X$ finite for all $g \in M$ implies that y is algebraically independent over $R \subset M$. Now define $\gamma_n = \sum_{i=0}^{n-1}(i!)x_{\alpha_i}$ and $y_n = \frac{y-\gamma_n}{n!} \in \widehat{P}$. Define $M' = (M + \sum_n y_n\mathbb{Z})R[y]$. Note that $ny_n = y_{n-1} - x_n$ and therefore $y_k \in y_\ell\mathbb{Z} + P$ for any $k \leq \ell < \omega$. Let $h \in M'$, then $h = \sum_{j=0}^n (m_j + y_{n_0}z_j)r_jy^j$ for some $m_j \in M$, $n_0 \in \mathbb{N}$, $z_j \in \mathbb{Z}$, $r_j \in R$. Thus $(n_0!)h = \sum_{j=0}^n \widetilde{m}_jy^j + z_nr_ny^{n+1} \in M[y]$, a 'polynomial' in y with coefficients in M. Since M/R is torsion we infer that $kh \in R[y]$, the ring of polynomials over R, for some $k \in \mathbb{N}$. It is now easy to see that M is pure in M'. Assume that $k\varphi \notin R$ for any $0 \notin k \in \mathbb{N}$ but $\varphi(y) \in M'$. Then there exists some $k_1 \in \mathbb{N}$

such that $k_1\varphi(y) = \sum_{j=0}^{n_1} r_j y^j$ for some $r_j \in R$. We distinguish between two cases:

CASE 1: $n_1 \leq 1$. Since $k\varphi \notin R$ for any $0 \notin k \in \mathbb{N}$ we have that $(k_1\varphi - r_1)(P) \neq \{0\}$ and we have some $b \in P$ such that $0 \neq (k_1\varphi - r_1)(b) \in M$. Since M_* is cotorsion-free, there is some $\mathbb{Z}$-adic number π such that $(k_1\varphi - r_1)(b\pi) \notin M_*$. Let $z = b\pi + y$ and assume that $\varphi(z) \in M'' = (M + \sum_{n<\omega} z_n\mathbb{Z})R[z]$ where $z_n = \frac{z-\gamma_n - b\pi_n}{n!}$ and $\pi_n \in \mathbb{Z}$ converge to π. This implies that for some $k_2 \in \mathbb{N}$ we have $k_2\varphi(z) = k_2\varphi(y + b\pi) = \sum_{j=0}^{n_2} r'_j z^j$. It follows that $k_2k_1(\varphi(z) - \varphi(y)) = k_1 \sum_{j=0}^{n_2} r'_j z^j - k_2(r_0 + r_1 y)$. We infer that $n_2 = 1$ and

$$
\begin{aligned}
k_2k_1\varphi(b\pi) = k_2k_1(\varphi(z) - \varphi(y)) &= k_1(r'_0 + r'_1 z) - k_2(r_0 + r_1 y) \\
&= k_1r'_0 - k_2r_0 + k_1r'_1 b\pi + (k_1r'_1 - k_2r_1)y \in M\pi.
\end{aligned}
$$

By supports, it follows that $k_1r'_1 - k_2r_1 = 0$ and thus $k_2(k_1\varphi(b) - r_1b)\pi = k_2k_1\varphi(b\pi) - k_2r_1b\pi = k_2k_1\varphi(b\pi) - k_1r'_1b\pi = k_1r'_0 - k_2r_0 \in M$. We obtain $(k_1\varphi(b) - r_1b)\pi \in M_*$, a contradiction.

If $n\varphi = r \in R$, then $n\varphi(1) = r \in R$. Since $S \subseteq R \subseteq \widehat{S}$, S is dense in R and there is some $s \in S$ such that $r = s + nt$ for some $t \in R$. Then $\varphi = r/n = (s + nt)/n = s/n + t$ and it follows that $\varphi - t = s/n \in Hom(P, M)$ and we have $s \in S \cap M \subseteq S \cap n\widehat{S} = nS$ since S is pure in $\widehat{S}$. Thus $s = ns'$ for some $s' \in S$ and it follows that $\varphi = (s' + t) \in R$.

CASE 2: $n_1 \geq 2$. Recall that $k_1\varphi(y) = \sum_{j=0}^{n_1} r_j y^j$ for some $r_j \in R$ and we may assume $0 \neq r_{n_1} \in R$. By cotorsion-freeness, there exists $\pi \in \widehat{\mathbb{Z}}$ such that $n_1\pi r_{n_1} \notin M$. Now consider $z = y + 1\pi$ and assume $k_2\varphi(z) = k_2\varphi(y + \pi) = \sum_{j=0}^{n_2} r'_j z^j$. Then

$$
k_1k_2\varphi(1\pi) = k_1k_2(\varphi(z) - \varphi(y)) = k_1 \sum_{j=0}^{n_2} r'_j z^j - k_2 \sum_{j=0}^{n_1} r_j y^j \in \pi M.
$$

By support arguments, we infer $n_1 = n_2 =: n$ and $k_1r'_n = k_2r_n$. This implies that $k_1r'_n y^{n-1}\binom{n}{1}\pi + k_1r'_{n-1}y^{n-1} - k_2r_{n-1}y^{n-1} = 0$ since $n \geq 2$. It follows that $k_2r_n n\pi = k_2r_{n-1} - k_1r'_{n-1} = w \in R$. There exist some $t \in R, s \in S$ such that $w = s + k_2t$, which implies $k_2r_nn\pi = s + k_2t$ and $k_2(r_nn\pi - t) = s \in S \cap k_2\widehat{S} = k_2S$. This implies $r_nn\pi - t \in S$ and we have the contradiction $r_nn\pi \in R \subseteq M$ to the choice of π. This shows that in this case $\varphi(y) \notin M'$ or $\varphi(z) \notin M''$. ∎

We are now ready for our

Construction: Let $\{\varphi_\beta : \beta < \lambda\}$ be the family of canonical homomorphisms provided by the Black Box. For $\beta < \lambda$ let

$$P_\beta = domain(\varphi_\beta) = S[x_\alpha : \alpha \in [\varphi_\beta]_X].$$

Inductively define elements $y_\gamma \in \widehat{P_\gamma}$ and rings $R^\gamma = S[\{x_\alpha : \alpha < \|\varphi_\gamma\|\} \cup \{y_\beta : \beta \leq \gamma\}]$ and R^γ-modules M^γ such that $R^\gamma \subseteq M^\gamma \subseteq \mathbb{Q}R^\gamma \subseteq \widehat{S[x_\alpha : \alpha < \lambda]}$, such that for all $\gamma < \beta < \lambda$,

(1) $\|y_\gamma\| = \|P_\gamma\| (= \|\varphi_\gamma\|)$;
(2) $R^\beta = S[\{x_\alpha : \alpha < \lambda\} \cup \{y_\gamma : \gamma < \beta\} \subseteq M^\beta$ with M^β / R^β torsion.
(3) M^β is cotorsion-free.

Let $R^0 = S[x_\alpha : \alpha < \lambda]$. If $\beta < \lambda$ is a limit ordinal then $R^\beta = \bigcup_{\gamma<\beta} R^\gamma$ and $M^\beta = \bigcup_{\gamma<\beta} M^\gamma$. Suppose $R^\beta, \beta < \lambda$, has been constructed.

Case 1: φ_β: $P_\beta \to \widehat{P_\beta}$ satisfies $\varphi_\beta(P_\beta) \subseteq M^\beta$ and $\varphi_\beta \notin R^\beta$. Here we apply our Step Lemma and find an element $y = y_\beta \in \widehat{P_\beta}$ such that $R^{\beta+1} = R^\beta[y_\beta]$ is a polynomial ring and $M^\beta \subset M^{\beta+1}$ is an $R^{\beta+1}$-module such that $\varphi_\beta(y_\beta) \notin M^{\beta+1}$.

Case 2: φ_β: $P_\beta \to \widehat{P_\beta}$ satisfies $\varphi_\beta(P_\beta) \nsubseteq M^\beta$ or $\varphi_\beta \in R^\beta$. In this case we define $y = y_\beta$ as in the Step Lemma without putting any conditions on $\varphi_\beta(y_\beta)$.

Finally, set $\widetilde{R} = \bigcup_{\beta<\lambda} R^\beta$ and $\widetilde{M} = \bigcup_{\beta<\lambda} M^\beta$.

Just like in [10, Section 2], it follows that $End(\widetilde{M}) = \widetilde{R}$ and $\widetilde{R} \subseteq \widetilde{M} \subseteq \mathbb{Q}\widetilde{R}$. Moreover, $\widetilde{R}$ is isomorphic to a polynomial ring over S in λ-many variables, i.e. the set $\{x_\alpha : \alpha < \lambda\} \cup \{y_\beta : \beta \in E^\#\}$ for some subset $E^\#$ of λ. After completing the remaining steps of the proof following [10], we will have:

Theorem 7: *Let S, κ, μ, λ be as in our Notation and $R = S[t_\alpha : \alpha < \lambda]$ be the polynomial ring over S in λ-many variables. Then there exists an R-module M such that $R \subseteq M \subseteq \mathbb{Q}R$, M/R is torsion and $End_{\mathbb{Z}}(M) = R$, i.e. M is a quasi-localization of $\mathbb{Z}$.*

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